Peertechz

MATHEMATICS

Computational Mathematics and its Applications

OPEN ACCESS JOURNAL

DOI: https://dx.doi.org/10.17352/cma

Research Article

Study on generalized BK-5th recurrent Finsler space

Adel Mohammed Ali Al-Qashbari[1]* and Saeedah Mohammed Saleh Baleedi[2]

[1]Deparment of Mathematics, Faculty of Education, Aden, University of Aden, Khormaksar, Aden, Yemen

[2]Deparment of Mathematics, Faculty of Education, Abyan, University of Abyan, Zingibar, Abyan, Yemen

Received: 21 November, 2023
Accepted: 06 December, 2023
Published: 07 December, 2023

***Corresponding author:** Adel Mohammed Ali Al-Qashbari, Deparment of Mathematics, Faculty of Education, Aden, University of Aden, Khormaksar, Aden, Yemen, Tel: 733678130; E-mail: Adel_ma71@yahoo.com

Keywords: N-dimensional finsler space F_n; generalized BK-5th recurrent spaces, Berwald's fifth-order covariant derivative, K^i_{jkh} Cartan's fourth curvature tensor and R^i_{jkh} Cartan's third curvature tensor

Abstract

In this paper, we present a novel new class and investigate the connection between the K-projective curvature tensor and other tensors of Finsler space F_n, this space is characterized by the property for Cartan's 4th curvature tensor K^i_{jkh} satisfies the certain relationship with the given covariant vectors field, we define this space as a generalized BK-5th recurrent space and denote it briefly by $GBK\text{-}5RF_n$. This paper aims to derive the fifth-order Berwald covariant derivatives of the torsion tensor H^i_{kh} and the deviation tensor H^i_h. Additionally, it demonstrates that the curvature vector K_j, the curvature vector H_k, and the curvature scalar H are all non-vanishing within the considered space. We have identified tensors that exhibit self-similarity under specific conditions. Furthermore, we have established the necessary and sufficient conditions for certain tensors in this space to have equal fifth-order Berwald covariant derivatives with their lower-order counterparts.

1. Introduction

In 1973, Sinha and Singh studied the properties of recurrent tensors in recurrent Finsler space [1]. Several works on recurrent Finsler space were done in the years 1973 and 1987. Verma (1991) discussed the recurrence property of Cartan's third curvature tensor R^i_{jkh} [2]. Dikshit (1992) discussed the bi-recurrence of Berwald curvature tensor H^i_{jkh} [3]. Qasem (2000) introduced and studied the recurrence conditions of the curvature tensor U^i_{jkh} in the sense of Berwald [4]. Qasem and Abdallah (2016) defined a generalized BR-recurrent Finsler space and obtained the necessary and sufficient conditions for the Berwald curvature tensor and Cartan's fourth curvature tensorto be generalized recurrent [5]. The generalized BK - recurrent Finsler space was introduced by Qasem and Baleedi (2016) this space whose Cartan's fourth curvature tensor K^i_{jkh} satisfies a recurrence relation [6]. They showed that the K-Ricci tensor, the curvature vector, and the curvature scalar are non-vanishing in the BK-recurrent Finsler space. Al-Qashbari and Qasem (2017) [7] Studied on generalized BR-Trirecurrent Finsler Space. In 2020, Al-Qashbari [8] introduced some identities for generalized curvature tensors in B-recurrent Finsler space. Bidabad and Sepasi, [9] completed Finsler spaces of constant negative Ricci curvature. Abu-Donia, Shenawy, and Abdehameed [10] studied the W^*-Curvature Tensor on Relativistic Space- times. Verstraelen [11] established a new submanifolds theory a contemplation of submanifolds in geometry of submanifolds. Deszcz, Głogowska, and Zafindratafa [12] established some conditions on hypersurfaces. In 2021, Opondo [13] studied the decomposition of Weyl projective curvature tensor

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized BK-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

in recurrent and bi-recurrent Finsler space. Chen [14] developmental Wintgen inequality and Wintgen ideal submanifolds. Eyasmin [15] studied hypersurfaces in a conformally flat space. Deszcz and Hotloś [16] defined and studied the geodesic mappings in a particular class of roter spaces. Deszcz, M. Głogowska, and M. Hotloś[17] studied the OpozdaVerstraelen affine curvature tensor on hypersurfaces. Decu, Deszcz, and Haesen [18] studied the classification of Roter-type spacetimes. In 2022, Deszcz, Głogowska, Hotloś, and Sawicz [19] studied the particular Roter-type equation on hypersurfaces in space forms. Derdzinski and Terek [20] introduced new examples of compact Weyl-parallel manifolds. Also they [21] studied the topology of compact rank-one ECSmanifolds. In 2023, Al-Qashbari and Al-Maisary [22] studied generalized *BW*- fourth recurrent in Finsler space. Shaikh, Hul, Datta, and Sakar [23] established new relations between the Kulkarni-Nomizu product of two (0,2) type tensors and the curvature tensors of type (0,4). Ali, Salman, Rahaman, and Pundeer, [24] obtained some properties of M-projective curvature tensor in spacetime.

Delving into the properties of an n-dimensional Finsler space F_n, we assume that its metric function F adheres to the established conditions outlined in the works of Deszcz, M. Głogowska, and M. Hotloś [17].

Positively homogeneous: $F\left(x, ky\right) = k\,F(x, y), k > 0.$

Positively: $F\left(x, y\right) > 0,\ y \neq 0.$

$\{\dot{\partial}_i\,\dot{\partial}_j F^2\left(x, y\right)\}\,\xi^i\,\xi^j, \dot{\partial}_i = \dfrac{\partial}{\partial y^i}$, is the positive definite $\forall$ variables ξ^i.

The corresponding metric tenser denoted by g_{ij}, the connection coefficients of Cartan represented by Γ^{*i}_{jk} and the connection coefficients of Berwald designated by G^i_{jk}, are all related to the metric function F.

$$\text{a) } g_{ij}\,y^i\,y^j = F^2,\ \text{b) } g_{ij}\,y^j = y_i,$$

$$\text{c) } g_{ij} = \frac{1}{2}\dot{\partial}_i y_j,\ \text{d) } y_i\,y^i = F^2,$$

$$\text{e) } g_{ij}\,g^{ik} = \delta^k_j = \begin{cases} 1 & if\ \ j = k\ , \\ 0 & if\ \ j \neq k\ . \end{cases}\ \text{f) } \delta^i_h\,g_{ik} = g_{hk}, \tag{1.1}$$

$$\text{g) } \delta^i_k\,y^k = y^i \text{ and h) } \delta^i_i = n\ .$$

The torsion tensor C_{ijk} defined by [12]

$$C_{ijk} = \frac{1}{2}\,\dot{\partial}_i g_{jk} = \frac{1}{4}\,\dot{\partial}_i\,\dot{\partial}_j\dot{\partial}_k\,F^2 \tag{1.2}$$

and its associate is the torsion tensor C^i_{jk} and it is defined by

$$\text{a) } C^h_{ik} = g^{hj}\,C_{ijk} \text{ and b) } C^i_{jk}\,y^k = C^i_{kj}\,y^k = 0. \tag{1.3}$$

These tensors satisfy the following conditions

$$\text{a) } C_{ijk}\,y^k = C_{kij}\,y^k = C_{jki}\,y^k = 0\ ;$$

$$\text{b) } G^i_{jkh}\,y^j = G^i_{hjk}\,y^j = G^i_{khj}\,y^j = 0\ ;$$

$$\text{c) } \delta^i_k\,C_{jin} = C_{jkn}\ ;$$

$$\text{d) } C_{jkr}\,g^{jk} = C_r$$

and

$$\text{e) } \Gamma^{*i}_{jkh}\,y^h = G^i_{jkh}\,y^h = 0\ ;$$

Where $G^i_{jkh} = \dot{\partial}_j\,G^i_{kh}$ and $\dot{\partial}_i = \dfrac{\partial}{\partial y^h}$. (1.4)

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized *BK*-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

The Berwald covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is defined as:

$$\mathcal{B}_k T^i_j = \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}. \tag{1.5}$$

The Berwald covariant derivatives of the metric function F, the vectors y^i, y_i and the unit vector l^i are all identically zero [11]. In other words,

$$\text{a) } \mathcal{B}_k F = 0 \text{, b) } \mathcal{B}_k y^i = 0, \text{ c) } \mathcal{B}_k y_i = 0 \text{ and d) } \mathcal{B}_k l^i = 0. \tag{1.6}$$

However, Berwald's covariant derivative of the metric tensor g_{ij} is not identically zero, meaning $\mathcal{B}_k g_{ij} \neq 0$. It is expressed as:

$$B_k g_{ij} = -2y^h B_h C_{ijk} = -2\, C_{ijk/h} y^h. \tag{1.7}$$

The covariant differential operator of Berwald with respect to x^h and the partial differential operator with respect to y^k commute, as defined by

$$\left(\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k\right) T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}, \tag{1.8}$$

Where T^i_j is any arbitrary tensor.

The second Berwald covariant derivative of the vector field X^i, with respect to X^k and X^h is given by:

$$\mathcal{B}_k \mathcal{B}_h X^i = \dot{\partial}_k \mathcal{B}_h X^i - \left(\dot{\partial}_s \mathcal{B}_h X^i\right) G^s_k + \left(\mathcal{B}_h X^r\right) G^i_{rk} - \left(\mathcal{B}_r X^i\right) G^r_{hk}. \tag{1.9}$$

The tensors K^i_{jkh} and R^i_{jkh} defined by

$$\text{a) } K^i_{jkh} = \partial_k \Gamma^{*i}_{hj} + \left(\partial_s \Gamma^{*i}_{jk}\right)\Gamma^{*s}_{th} y^t + \Gamma^{*i}_{mk} \tilde{A}^{*m}_{hj} - \partial_h \Gamma^{*i}_{kj} - \left(\overset{\ddot{y}}{\partial}_s \Gamma^{*i}_{jh}\right)\Gamma^{*s}_{tk} y^t - \Gamma^{*i}_{mh}\Gamma^{*m}_{kj}$$

and

$$\text{b) } R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + \left(\dot{\partial}_r \Gamma^{*i}_{jk}\right)\Gamma^{*r}_{sh} y^s + C^i_{jm}\left(\partial_k \Gamma^{*m}_{sh} y^s - \Gamma^{*m}_{kr}\Gamma^{*r}_{sh} y^s\right) + \Gamma^{*i}_{mk}\Gamma^{*m}_{jh} - \partial_k \Gamma^{*i}_{jh} - \left(\dot{\partial}_r \Gamma^{*i}_{jh}\right)\Gamma^{*r}_{sk} y^s - C^i_{jm}\left(\partial_h \Gamma^{*m}_{sk} y^s - \Gamma^{*m}_{hr}\Gamma^{*r}_{sk} y^s\right) - \Gamma^{*i}_{mh}\Gamma^{*m}_{jk}. \tag{1.10}$$

The aforementioned tensors, denoted as Cartan's fourth curvature tensor and Cartan's third curvature tensor, respectively, exhibit skew-symmetry with respect to their last two lower indices and positive homogeneity of degree zero in their directional arguments. These tensors adhere to the following relations:

$$\text{a) } R^i_{jkh} y^j = K^i_{jkh} y^j = H^i_{kh} \text{, b) } H^i_{jkh} = K^i_{jkh} + y^m (\dot{\partial}_j K^i_{mkh}),$$
$$\text{c) } K^i_{jkh} = R^i_{jkh} - C^i_{js} H^s_{kh} \text{, d) } H^i_{jkh} - K^i_{jkh} = P^i_{jkh} + P^r_{jk} P^i_{rh} - P^i_{jhk} - P^r_{jh} P^i_{rk},$$
$$\text{e) } R_{ijhk} = K_{ijhk} + C_{ijm} K^s_{rhk} y^m \text{ and f) } R_{ijkh} = g_{rj} R^r_{ikh}. \tag{1.11}$$

Ricci tensor K_{jk}, curvature vector K_j and curvature scalar K derived from the curvature tensor K^i_{jkh} are defined as:

$$\text{a) } K^i_{jki} = K_{jk} \text{, b) } K_{jk} y^k = K_j \text{ and c) } K_{jk} g^{jk} = K. \tag{1.12}$$

Ricci tensor R_{jk}, the deviation tensor R^i_h and curvature scalar R derived from the curvature tensor R^i_{jkh} are defined as:

$$\text{a) } R^r_{jkr} = R_{jk} \text{, b) } R^i_{jkh} g^{jk} = R^i_h \text{ and c) } R_{jk} g^{jk} = R. \tag{1.13}$$

The curvature tensor of Berwald H^i_{jkh}, torsion tensor H^i_{kh}, Ricci tensor H_{jk}, deviation tensor H^i_h and curvature scalar H is defined as

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized BK-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

$$a)H^i_{jkh}\, y^j ,= H^i_{kh} b)H^i_{kh}\, y^k = H^i_h, c)H^i_k\, y^k = 0, d)H_{jk} = H^r_{jkr}$$
$$e)H^r_{kr} = H_k, f)H^i_{jk}\, y_i = 0, g)y_i H^i_k = 0, \tag{1.14}$$
$$h)K_{jk}\, y^j = H_k \text{ and } i)K_j y^j = (n-1)H.$$

A Finsler space in which the Berwald connection parameter G^i_{kh} does not depend on the directional coefficients y^i is known as an affinely connected space (Berwald space) [17].

Therefore, an affinely connected space is defined by one of the equivalent conditions

$$a)\mathcal{B}_k g_{ij} = 0 \text{ and } b)\mathcal{B}_k g^{ij} = 0 \tag{1.15}$$

2. A Generalized *BK*-5th recurrent Finsler space

Let us explore in $GBK\text{-}RF_n$ for which whose Cartan's fourth curvature tensor K^i_{jkh} is defined as [9]:

$$\mathcal{B}_m K^i_{jkh} = a_m K^i_{jkh} + b_m \left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right), K^i_{jkh} \neq 0$$

Is called generalized BK-recurrent space, where B_m is a covariant derivative of the first order (Berwald's covariant differential operator) with respect to x^m. Taking the covariant derivative of the fifth order for the above equation in the sense of Berwald with respect to x^l, x^m, x^n, x^q *and* x^s respectively, we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K^i_{jkh} = a_{sqlnm} K^i_{jkh} + b_{sqlnm}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$
$$-c_{sqlnm}\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - d_{sqlnm}\left(\delta^i_h C_{jkl} - \delta^i_k C_{jhl}\right)$$
$$-e_{sqlnm}\left(\delta^i_h C_{jkq} - \delta^i_k C_{jhq}\right) - 2b_{qlnm} y^r \mathcal{B}_r \left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right). \tag{2.1}$$

Multiplying (2.1) by y^j, using (1.6b), (1.11a), (1.4a), and (1.1b), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{sqlnm} H^i_{kh} + b_{sqlnm}\left(\delta^i_h y_k - \delta^i_k y_h\right). \tag{2.2}$$

Multiplying (2.2) by y^k, using (1.6b), (1.14b), (1.1d), and (1.1g), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h = a_{sqlnm} H^i_h + b_{sqlnm}\left(\delta^i_h F^2 - y^i y_h\right). \tag{2.3}$$

In conclusion, we find that

Theorem 2.1: In the $GBK\text{-}5RF_n$, Berwald's covariant derivatives of the fifth order for the torsion tensor H^i_{kh} and the deviation tensor H^i_h are given by the conditions (2.2) and (2.3), respectively.

Summation over the indices i and h in condition (2.1), using (1.12a), (1.4c), (1.1f) and (1.1h), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K_{jk} = a_{sqlnm} K_{jk} + b_{sqlnm}(n-1) g_{jk}$$
$$-c_{sqlnm}(n-1) C_{jkn} - d_{sqlnm}(n-1) C_{jkl}$$
$$-e_{sqlnm}(n-1) C_{jkq} - 2b_{qlnm} y^r \mathcal{B}_r (n-1) C_{jks}. \tag{2.4}$$

Multiplying (2.4) by y^k, using (1.6b), (1.12b), (1.4a) and (1.1b), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K_j = a_{sqlnm} K_j + b_{sqlnm}(n-1) y_j. \tag{2.5}$$

Multiplying (2.5) by y^j, using (1.6b), (1.14i) and (1.1d), we obtain

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized *BK*-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H = a_{sqlnm} H + b_{sqlnm} F^2. \quad (2.6)$$

Multiplying (2.4) by y, using (1.6b), (1.14h), (1.4a) and (1.1b), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H_k = a_{sqlnm} H_k + b_{sqlnm}\left(n-1\right) y_k. \quad (2.7)$$

In conclusion, we find that

Theorem 2.2: In the $GBK\text{-}5RF_n$, the curvature vector k_j, the curvature vector H_k and the curvature scalar H are all nonzero.

3. Necessary and sufficient condition

Let us explore an $GBK\text{-}5RF_n$ which is characterized by the condition (2.1).

Multiplying (2.4) by g^{jk}, and let a Berwald space (affinely connected space) and using (1.12c), (1.15b), (1.1e), (1.1h), and (1.4d), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K = a_{sqlnm} K + n\left(n-1\right) b_{sqlnm} \quad (3.1)$$

$$-\left(n-1\right)\left[c_{sqlnm} c_n + d_{sqlnm} c_l + e_{sqlnm} c_q + 2b_{qlnm} y^r \mathcal{B}_r C_s\right]$$

We can express the above equation in different ways as

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K = a_{sqlnm} K \quad (3.2)$$

If and only if

$$nb_{sqlnm} - \left[c_{sqlnm} c_n + d_{sqlnm} c_l + e_{sqlnm} c_q + 2b_{qlnm} y^r \mathcal{B}_r C_s\right] = 0 \quad (3.3)$$

In conclusion, we find that

Theorem 3.1: In the $GBK\text{-}5RF_n$, (as defined by Berwald space), the fifth-order Berwald covariant derivative of the curvature scalar K is directly proportional to the curvature scalar itself solely under the condition that equation (3.3) is valid.

On account of (2.4), we have

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K_{jk} = a_{sqlnm} K_{jk}. \quad (3.4)$$

If and only if

$$b_{sqlnm} g_{jk} - c_{sqlnm} C_{jkn} - d_{sqlnm} C_{jkl} - e_{sqlnm} C_{jkq} - 2b_{qlnm} y^r \mathcal{B}_r C_{jks} = 0 \quad (3.5)$$

In conclusion, we find that

Theorem 3.2: In the $GBK\text{-}5RF_n$, covariant derivative of Berwald on the fifth order for Ricci tensor K_{jk} is proportional to the tensor itself if and only if (3.5) is valid.

On account of the condition [15]:

$$\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m (\delta^i_h y_k - \delta^i_k y_h) \quad (3.6)$$

And in view the condition (2.2), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = \mathcal{B}_m H^i_{kh} \quad (3.7)$$

If and only if

$$\lambda_m H^i_{kh} + \mu_m (\delta^i_h y_k - \delta^i_k y_h) = a_{sqlnm} H^i_{kh} + b_{sqlnm}\left(\delta^i_h y_k - \delta^i_k y_h\right) \quad (3.8)$$

In conclusion, we find that

Theorem 3.3: In the $GBK\text{-}5RF_n$, covariant derivative of Berwald on the first order and fifth order for the torsion tensor H^i_{kh} both are equal if and only if (3.8) is valid.

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized *BK*-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

On account of the condition [15]:

$$\mathcal{B}_m H^i_h = \lambda_m H^i_h + \mu_m (\delta^i_h F^2 - y^i y_h). \tag{3.9}$$

And in view the condition (2.3), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h = \mathcal{B}_m H^i_h. \tag{3.10}$$

If and only if

$$\lambda_m H^i_h + \mu_m \left(\delta^i_h F^2 - y^i y_h\right) = a_{sqlnm} H^i_h + b_{sqlnm} \left(\delta^i_h F^2 - y^i y_h\right). \tag{3.11}$$

In conclusion, we find that

Theorem 3.4: In the $GBK\text{-}5RF_n$, covariant derivative of Berwald on the first order and fifth order for the deviation tensor H^i_h both are equal if and only if (3.11) is valid.

On account of the condition [8]:

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{kh} = a_{lmns} W^i_{kh} + b_{lmns} \left(\delta^i_h y_k - \delta^i_k y_h\right). \tag{3.12}$$

And in view the condition (2.2), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{kh} \tag{3.13}$$

If and only if

$$a_{sqlnm} H^i_{kh} + b_{sqlnm} \left(\delta^i_h y_k - \delta^i_k y_h\right) = a_{lmns} W^i_{kh} + b_{lmns} \left(\delta^i_h y_k - \delta^i_k y_h\right). \tag{3.14}$$

In conclusion, we find that

Theorem 3.5: In the $GBK\text{-}5RF_n$, covariant derivative of Berwald on the fifth order for the torsion tensor H^i_{kh} and of the fourth order for projective torsion tensor W^i_{kh} both are equal if and only if (3.14) are valid.

On account of the condition [8]:

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_h = a_{lmns} W^i_h + b_{lmns} (\delta^i_h F^2 - y^i y_h). \tag{3.15}$$

And in view the condition (2.3), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h = \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_h. \tag{3.16}$$

If and only if

$$a_{sqlnm} H^i_h + b_{sqlnm} \left(\delta^i_h F^2 - y^i y_h\right) = a_{lmns} W^i_h + b_{lmns} (\delta^i_h F^2 - y^i y_h) \tag{3.17}$$

In conclusion, we find that

Theorem 3.6: In the $GBK\text{-}5RF_n$, covariant derivative of Berwald on the fifth order for the deviation tensor H^i_h and of the fourth order for projective deviation tensor W^i_h both are equal if and only if (3.17) holds good.

On account of the condition [8]:

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W_k = a_{lmns} W_k + (n-1) b_{lmns} y_k \tag{3.18}$$

and

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W = a_{lmns} W + (n-1) b_{lmns} F^2 \tag{3.19}$$

With the condition (2.7) and (2.6), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H_k = \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W_k \tag{3.20}$$

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized *BK*-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

If and only if

$$a_{sqlnm}H_k + b_{sqlnm}\left(n-1\right)y_k = a_{lmns}W_k + (n-1)b_{lmns}y_k \tag{3.21}$$

And

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H = \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W \tag{3.22}$$

If and only if

$$a_{sqlnm}H + b_{sqlnm}F^2 = a_{lmns}W - \left(n-1\right)b_{lmns}F^2 \tag{3.23}$$

In conclusion, we find that

Theorem 3.7: In the GBK-$5RF_n$, covariant derivative of Berwald on the fifth order for curvature vector H_K and curvature scalar H, both are equal to Berwald's covariant derivative of the fifth order for the curvature vector W_K and curvature scalar W, respectively if and only if (3.21) and (3.23), respectively hold good.

On account of the condition [4]:

$$\mathcal{B}_n\mathcal{B}_m P^i_{kh} = a_{mn}P^i_{kh} + b_{mn}(\delta^i_h y_k - \delta^i_k y_h) \tag{3.24}$$

And the condition (2.2), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = \mathcal{B}_n\mathcal{B}_m P^i_{kh} \tag{3.25}$$

If and only if

$$a_{sqlnm}H^i_{kh} + b_{sqlnm}\left(\delta^i_h y_k - \delta^i_k y_h\right) = a_{mn}P^i_{kh} + b_{mn}\left(\delta^i_h y_k - \delta^i_k y_h\right) \tag{3.26}$$

In conclusion, we find that

Theorem 3.8: In the GBK-$5RF_n$, covariant derivative of Berwald on the fifth order for torsion tensor H^i_{kh} and of the second order for torsion tensor P^i_{kh} both are equal if and only if (3.26) holds good.

On account of the condition [4]:

$$\mathcal{B}_n\mathcal{B}_m P_k = a_{mn}P_k + b_{mn}\left(n-1\right)y_k \tag{3.27}$$

and in view the condition (2.7), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H_k = \mathcal{B}_n\mathcal{B}_m P_k \tag{3.28}$$

If and only if

$$a_{sqlnm}H_k + b_{sqlnm}\left(n-1\right)y_k = a_{mn}P_k + b_{mn}(n-1)y_k \tag{3.29}$$

In conclusion, we find that

Theorem 3.9: In the GBK-$5RF_n$, covariant derivative of Berwald on of the fifth order for the curvature vector H_K and second order for the curvature vector $P_{k,}$ both are equal if and only if (3.29) holds good.

Using (1.11c) in (2.1), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = a_{sqlnm}R^i_{jkh} + b_{sqlnm}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) \tag{3.30}$$

$$-c_{sqlnm}\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - d_{sqlnm}\left(\delta^i_h C_{jkl} - \delta^i_k C_{jhl}\right)$$

$$-e_{sqlnm}\left(\delta^i_h C_{jkq} - \delta^i_k C_{jhq}\right) - 2b_{qlnm}y^r\mathcal{B}_r\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right)$$

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized *BK*-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

$$+\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^i_{jt}H^t_{kh})-a_{sqlnm}(C^i_{jt}H^t_{kh}).$$

This exhibits that

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}=a_{sqlnm}R^i_{jkh}+b_{sqlnm}\left(\delta^i_h g_{jk}-\delta^i_k g_{jh}\right)$$

$$-c_{sqlnm}\left(\delta^i_h C_{jkn}-\delta^i_k C_{jhn}\right)-d_{sqlnm}\left(\delta^i_h C_{jkl}-\delta^i_k C_{jhl}\right)$$

$$-e_{sqlnm}\left(\delta^i_h C_{jkq}-\delta^i_k C_{jhq}\right)-2b_{qlnm}y^r\mathcal{B}_r\left(\delta^i_h C_{jks}-\delta^i_k C_{jhs}\right) \quad (3.31)$$

If and only if

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^i_{jt}H^t_{kh})=a_{sqlnm}(C^i_{jt}H^t_{kh}) \quad (3.32)$$

In conclusion, we find that

Theorem 3.10: In the $GBK\text{-}5RF_n$, Cartan's third curvature tensor R^i_{jkh} is $GBK\text{-}5RF_n$ if and only if (3.32) it holds good.

Multiplying (3.30) by g_{if}, and let the space be a Berwald space and using (1.15a), (1.11f), and (1.1f), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{jfkh}=a_{sqlnm}R_{jfkh}+b_{sqlnm}\left(g_{hf}g_{jk}-g_{kf}g_{jh}\right)$$

$$-2b_{qlnm}y^r\mathcal{B}_r\left(g_{hf}C_{jks}-g_{kf}C_{jhs}\right)-c_{sqlnm}\left(g_{hf}C_{jkn}-g_{kf}C_{jhn}\right)$$

$$-d_{sqlnm}\left(g_{hf}C_{jkl}-g_{kf}C_{jhl}\right)-e_{sqlnm}\left(g_{hf}C_{jkq}-g_{kf}C_{jhq}\right)$$

$$+\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C_{jft}H^t_{kh})-a_{sqlnm}(C_{jft}H^t_{kh}) \quad (3.33)$$

This exhibits that

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{jfkh}=a_{sqlnm}R_{jfkh} \quad (3.34)$$

If and only if

$$b_{sqlnm}\left(g_{hf}g_{jk}-g_{kf}g_{jh}\right)-2b_{qlnm}y^r\mathcal{B}_r\left(g_{hf}C_{jks}-g_{kf}C_{jhs}\right) \quad (3.35)$$

$$-c_{sqlnm}\left(g_{hf}C_{jkn}-g_{kf}C_{jhn}\right)-d_{sqlnm}\left(g_{hf}C_{jkl}-g_{kf}C_{jhl}\right)$$

$$-e_{sqlnm}\left(g_{hf}C_{jkq}-g_{kf}C_{jhq}\right)+\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C_{jft}H^t_{kh})$$

$$-a_{sqlnm}\left(C_{jft}H^t_{kh}\right)=0$$

Summation over the indices i and h in condition (3.30), using (1.13a), (1.1h), (1.1f) and (1.14c), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{jk}=a_{sqlnm}R_{jk}+b_{sqlnm}\left(n-1\right)g_{jk}-2b_{qlnm}y^r\mathcal{B}_r\left(n-1\right)C_{jks}$$

$$-c_{sqlnm}\left(n-1\right)C_{jkn}-d_{sqlnm}\left(n-1\right)C_{jkl}-e_{sqlnm}\left(n-1\right)C_{jkq} \quad (3.36)$$

$$+\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^u_{jt}H^t_{ku})-a_{sqlnm}(C^u_{jt}H^t_{ku})$$

This exhibits that

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{jk}=a_{sqlnm}R_{jk} \quad (3.37)$$

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized *BK*-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

If and only if

$$b_{sqlnm}(n-1)g_{jk} - 2b_{qlnm}y^r \mathcal{B}_r (n-1)C_{jks} - c_{sqlnm}(n-1)C_{jkn}$$

$$-d_{sqlnm}(n-1)C_{jkl} - e_{sqlnm}(n-1)C_{jkq} + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^u_{jt}H^t_{ku})$$

$$-a_{sqlnm}(C^u_{jt}H^t_{ku}) = 0 \tag{3.38}$$

Multiplying (3.36) by g^{jk}, and let the space be a Berwald space and using (1.15b), (1.13c), (1.4d) and (1.1e), (1.1h), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R = a_{sqlnm}R + n(n-1)b_{sqlnm} - 2b_{qlnm}y^r\mathcal{B}_r(n-1)C_s$$

$$-c_{sqlnm}(n-1)C_n - d_{sqlnm}(n-1)C_l - e_{sqlnm}(n-1)C_q$$

$$+[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^u_{jt}H^t_{ku}) - a_{sqlnm}(C^u_{jt}H^t_{ku})]g^{jk} \tag{3.39}$$

This exhibits that

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R = a_{sqlnm}R \tag{3.40}$$

If and only if

$$b_{sqlnm} - 2b_{qlnm}y^r\mathcal{B}_r C_s - c_{sqlnm}C_n - d_{sqlnm}C_l - e_{sqlnm}C_q$$

$$+[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^u_{jt}H^t_{ku}) - a_{sqlnm}(C^u_{jt}H^t_{ku})]g^{jk} = 0 \tag{3.41}$$

In conclusion, we find that

Theorem 3.11: In the the $GBK\text{-}5RF_n$, in the sense of Berwald space the covariant derivative of Berwald on the fifth order for associate curvature tensor R_{ijkh} of the tensor R^i_{jkh} and the curvature scalar R all are proportional to the tensor itself if and only if (3.35) and (3.41), respectively hold good.

Theorem 3.12: In the $GBK\text{-}5RF_n$, (in the sense of Berwald space), the covariant derivative of Berwald on the fifth order for the Ricci tensor Rjk is proportional to the tensor itself if and only if (3.38) it holds good.

Using (1.11d) in (2.1), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l(H^i_{jkh} - P^i_{jkh} - P^r_{jk}P^i_{rh} + P^i_{jhk} + P^r_{jh}P^i_{rk})$$

$$= a_{sqlnm}(H^i_{jkh} - P^i_{jkh} - P^r_{jk}P^i_{rh} + P^i_{jhk} + P^r_{jh}P^i_{rk}) + b_{sqlnm}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$

$$-c_{sqlnm}(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) - d_{sqlnm}(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})$$

$$-e_{sqlnm}(\delta^i_h C_{jkq} - \delta^i_k C_{jhq}) - 2b_{qlnm}y^r\mathcal{B}_r(\delta^i_h C_{jks} - \delta^i_k C_{jhs}) \tag{3.42}$$

Summation over the indices i and h in condition (3.42), using (1.14d), (1.1h), (1.1f) and (1.4c), we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l(H_{jk} - P^t_{jkt} - P^r_{jk}P^t_{rt} + P^t_{jtk} + P^r_{jt}P^t_{rk})$$

$$= a_{sqlnm}(H_{jk} - P^t_{jkt} - P^r_{jk}P^t_{rt} + P^t_{jtk} + P^r_{jt}P^t_{rk}) + b_{sqlnm}(n-1)g_{jk}$$

$$-2b_{qlnm}y^r\mathcal{B}_r(n-1)C_{jks} - c_{sqlnm}(n-1)C_{jkn} - d_{sqlnm}(n-1)C_{jkl}$$

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized *BK*-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

$$-e_{sqlnm}\left(n-1\right)C_{jkq} \tag{3.43}$$

We can express the above equation in different ways as

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H_{jk} = a_{sqlnm} H_{jk} \tag{3.44}$$

If and only if

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(P^t_{jkt}+P^r_{jk}P^t_{rt}-P^t_{jtk}-P^r_{jt}P^t_{rk}\right)$$

$$+a_{sqlnm}(-P^t_{jkt}-P^r_{jk}P^t_{rt}+P^t_{jtk}+P^r_{jt}P^t_{rk})+b_{sqlnm}\left(n-1\right)g_{jk}$$

$$-2b_{qlnm}y^r\mathcal{B}_r\left(n-1\right)C_{jks}-c_{sqlnm}\left(n-1\right)C_{jkn}$$

$$-d_{sqlnm}\left(n-1\right)C_{jkl}-e_{sqlnm}\left(n-1\right)C_{jkq}=0 \tag{3.45}$$

In conclusion, we find that

Theorem 3.13: In the $GBK\text{-}5RF_n$, in the sense of Berwald space, the covariant derivative of Berwald on the fifth order for the Ricci tensor H_{jk} is proportional to the tensor itself if and only if (3.45) it holds good.

4. Composition relations between Cartan's third curvature tensor and conformal Curvature Tensor in *GBK-5RF$_n$*

In this section, we presented the relationship between Cartan's third curvature tensor R_{ijkh} and conformal curvature tensor C_{ijkh} in $GBK\text{-}5RF_n$

Definition 4.1: A conformal curvature tensor C_{ijkh} (also known as Weyl conformal curvature tensor) is defined as [2]:

$$R_{ijkh}=C_{ijkh}+\frac{1}{2}\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)+\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right) \tag{4.1}$$

Taking the covariant derivative of 5th order for (4.1) in the sense of Berwald, we obtain

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\ R_{ijkh}=\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh}+\frac{1}{2}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m$$

$$\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)+\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)\right] \tag{4.2}$$

Using (1.11f) and (1.15 a) in (4.2), we obtain

$$g_{rj}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\ R^r_{ikh}=\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh}+\frac{1}{2}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m$$

$$\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)+\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)\right] \tag{4.3}$$

We can express the above equation in different ways as

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\ R^r_{ikh}=\frac{1}{g_{rj}}\left[\begin{array}{l}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh}+\frac{1}{2}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)\\+\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)\right]\end{array}\right] \tag{4.4}$$

In conclusion, we find that

Theorem 4.1: In the $GBK\text{-}5RF_n$, (in the sense of Berwald space), the covariant derivative of Berwald on the fifth order for Cartan's

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized *BK*-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

third curvature tensor R^r_{ikh} and the conformal curvature tensor C_{ijkh}, linking together by the relation (4.4).

From (4.3), we have

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh} = g_{rj}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh} = g_{rj}\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}\right) \quad (4.5)$$

If and only if

$$\frac{1}{2}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right) + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{6}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\right] = 0 \quad (4.6)$$

In conclusion, we find that

Theorem 4.2: In the $GBK\text{-}5RF_n$, (in the sense of Berwald space), the covariant derivative of Berwald on the fifth order for the conformal curvature tensor C_{ijkh} is proportional to the Berwald covariant derivative of the Cartan's third curvature tensor R^r_{ikh} by (4.5) if and only if (4.6) it holds good.

Using (1. e 1), when $[i \neq h \neq k]$ in (3.31) and using it in (4.4), we obtain

$$R^r_{ikh} = \frac{1}{a_{sqlnm}\, g_{rj}}\left[\begin{array}{l}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh} + \frac{1}{2}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right)\\ +\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{6}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\right]\end{array}\right] \quad (4.7)$$

In conclusion, we find that

Theorem 4.3: In the $GBK\text{-}5RF_n$, the Cartan's third curvature tensor R^r_{ikh} and the Berwald covariant derivative of the fifth order for conformal curvature tensor C_{ijkh}, linking together by the relation (4.7).

The linking of Cartan's third curvature tensor with the conformal curvature tensor is a relationship between two tensors in differential geometry. The first tensor is Cartan's third curvature tensor which measures the local curvature of a metric space. The second tensor is the conformal curvature tensor which measures the local curvature of a transformed metric space. The relationship states that Cartan's third curvature tensor can be expressed as the product of the conformal curvature tensor and the metric factor.

The relationship linking Cartan's third curvature tensor with the conformal curvature tensor can be used to study the properties of transformed metric spaces. For example, it can be used to determine whether the metric space is connected or not.

Findings summary

In general relativity, the metric in Finsler space is defined as a function that depends on the velocity vector. A recurrent Finsler space is a Finsler space where the metric is symmetric around the velocity vector. The fifth order is the order of the velocity vector. In this context, (recurrent) refers to the fact that the metric depends on the velocity vector up to the fifth order. A generalized fifth-order recurrent Finsler space can be used to:

- Describe anti-desitter spacetime, where the metric is negative.
- Describe curved spacetime, where the metric depends on the spatial and temporal coordinates.
- Describe multi-dimensional spacetime, where there are more than three spatial dimensions.

Conclusion

A generalized fifth-order recurrent Finsler space is a new geometric structure with great potential. It can be used to describe a variety of geometric structures. Research on these new structures is still ongoing, but there are many potential applications for them. For example: developing new models of the universe, studying the properties of dark matter, and developing new technologies for space travel.

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized *BK*-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002

References

1. Sinha BB, Singh SP. Recurrent Finsler space of second order II. Indian Journal of Pure and Applied Mathematics. 1973; 4(1): 45-50.
2. Verma R. Some transformations in Finsler spaces. D. Phil. Thesis, University of Allahabad, (Allahabad) (India), 1991.
3. Dikshit S. Certain types of recurrences in Finsler spaces. D. Phil. Thesis, University of Allahabad, (Allahabad) (India). 1992.
4. Qasem FYA. On Transformation in Finsler Spaces. D.Phil Thesis University of Allahabad, (Allahabad) (India). 2000.
5. Qasem FYA, Abdallah AAA. On study generalized *BR* -recurrent Finsler space. International Journal of Mathematics and its Applications. 2016; 4: 113- 121.
6. Qasem FYA, Baleedi SMS. On a Generalized *BR*-Recurrent Finsler Space, International Journal of Science Basic and Applied Research. 2016; 28: 195-203.
7. AL-Qashbari AMA, Qasem FYA. Study on Generalized *BR* -Trirecurrent Finsler Space, Journal of Yemen Engineer, Faculty of Engineering, University of Aden. 2017; 15: 79-89.
8. AL-Qashbari AMA. On Generalized for Curvature Tensors P^{i}_{jkh} of Second Order in Finsler Space. Univ.Aden J Nat and Appl Sci. 2020; 24: 171-176.
9. Bidabad B, Sepasi M. Complete Finsler Spaces of Constant Negative Ricci Curvature. J of Math. D.G. 2020; 1:1–12.
10. Abu-Donia H, Shenawy S, Abdehameed A. The W*-Curvature Tensor on Relativistic Spacetimes, Kyungpook Mathematical Journal. 2020; 60:185-195.
11. Verstraelen L. Submanifolds theory–A contemplation of submanifolds, in: Geometry of Submanifolds, AMS Special Session on Geometry of Submanifolds in Honor of Bang- Yen Chen's 75th Birthday, October 20-21, 2018, University of Michigan Ann Arbor Michigan J Van der Veken, et al. (eds.), Contemporary Math. 756 Amer Math Soc. 2020; 21-56.
12. Deszcz R, Głogowska M, Zafindratafa G. Hypersurfaces in space forms satisfying some generalized Einstein metric condition. J Geom Phys. 2020; 148: 103562.
13. Opondo MA. Study of Projective curvature tensor K^{i}_{jkh} in bi-recurrent Finsler space, M. Sc. Dissertation, Kenyatta University, (Nairobi), (Kenya). 2021.
14. Chen BY. Recent developments in Wintgen inequality and Wintgen ideal submanifolds. Int Electron J Geom. 2021; 14: 1-40.
15. Eyasmin S. Hypersurfaces in a conformally flat space. Int J Geom Methods Modern Phys. 2021; 18: 2150067.
16. Deszcz R, Hotloś M. On geodesic mappings in a particular class of Roter spaces. Colloq Math. 2021; 166: 267-290.
17. Deszcz R, Głogowska M, Hotloś M. On hypersurfaces satisfying conditions determined by the Opozda Verstraelen affine curvature tensor. Ann Polon Math. 2021; 126: 215–240.
18. Decu S, Deszcz R, Haesen S. A classification of Roter type spacetimes. Int J Geom Meth Modern Phys.2021; 18: 2150147.
19. Deszcz R, Głogowska Hotloś M, Sawicz K. Hypersurfaces in space forms satisfying a particular Roter type equation. 2022; 28.
20. Derdzinski A, Terek I. New examples of compact Weyl-parallel manifolds. arXiv: 2210.03660v1 [math.DG] 7 Oct 2022; 11.
21. Derdzinski A, Terek I. The topology of compact rank-one ECS manifolds. arXiv: 2210.09195v1 [math.DG] 17 Oct 2022; 17.
22. AL-Qashbari AMA, AL-Maisary AAM. Study on Generalized of Fourth Order Recurrent in Finsler space. Journal of Yemen Engineer. 2023; 17: 1-13.
23. Shaikh AA, Hul SK, Datta BR, Sakar M. On Curvature Related Geometric Properties of Hayward Black Hole Spacetime, arXiv: 2303.00932v1[math.DG] 23 Feb(2023); 1-29.
24. Ali M, Salman M, Rahaman F, Pundeer N. On some properties of M-projective curvature tensor in spacetime of general relativity. arXiv:2209.12692v2 [gr-qc] 26 May (2023), 1-17.
25. Rund H. The differential geometry of Finsler spaces. Springer-Verlag, Berlin Göttingen- Heidelberg. 2nd Edit. (In Russian), Nauka, (Moscow). 1981.
26. Hadi WHA. Study of Certain Generalized Birecurrent in Finsler space. PhD. Thesis, University of Aden. (Yemen). 2016.
27. Emamian MH, Tayebi A. Generalized Douglas-Weyl Finsler Metrics, Iranian Journal of Mathematical Sciences and Informatics. 2015; 10: 67-75.
28. Baleedi SMS. On certain generalized *BK*-recurrent Finsler space, M. Sc. Dissertation. University of Aden. Aden Yemen. 2017.
29. Ahsan Z, Ali M. On some properties of *W* -curvature tensor. Palestine Journal of Mathematics. 2014; 3(1): 61-69.

Citation: Ali Al-Qashbari AM, Saleh Baleedi SM (2023) Study on generalized *BK*-5th recurrent Finsler space. Comput Math Appl. 1(1): 009-020.
DOI: https://dx.doi.org/10.17352/cma.000002